

## 4.4 | Uniform Circular Motion

### Learning Objectives

By the end of this section, you will be able to:

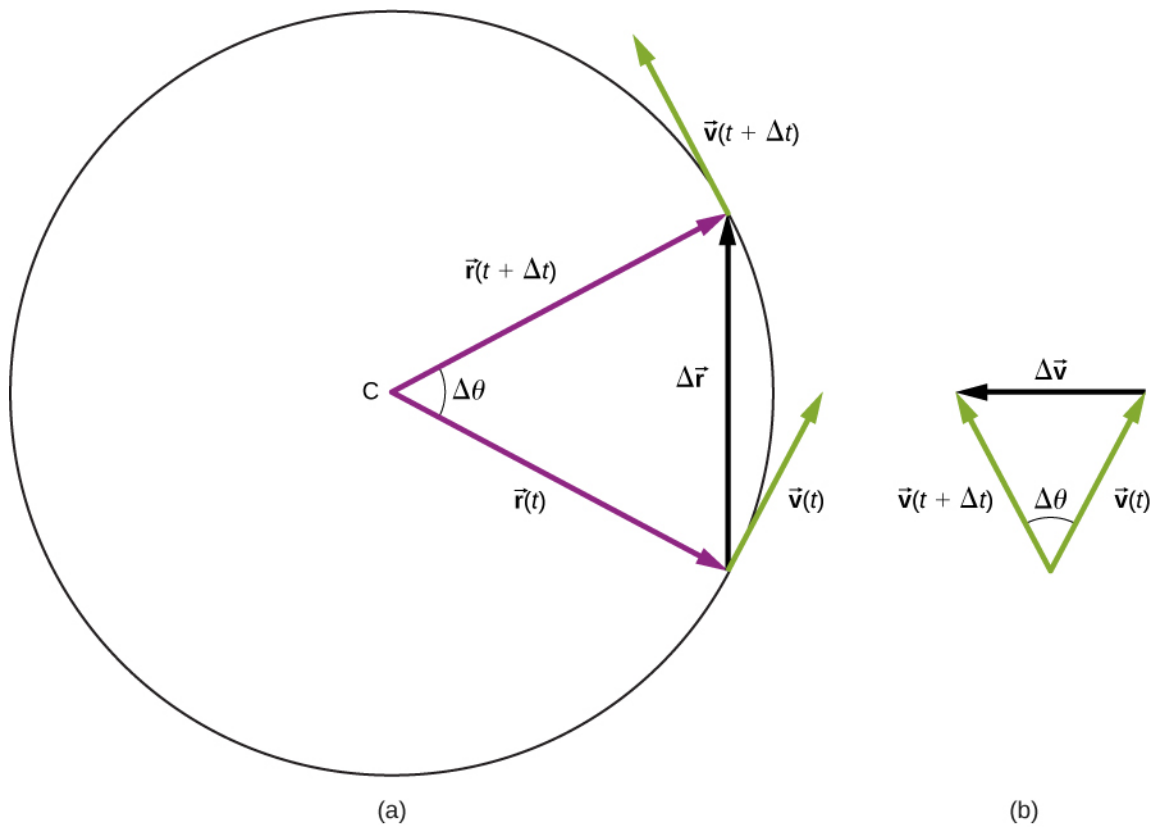
- Solve for the centripetal acceleration of an object moving on a circular path.
- Use the equations of circular motion to find the position, velocity, and acceleration of a particle executing circular motion.
- Explain the differences between centripetal acceleration and tangential acceleration resulting from nonuniform circular motion.
- Evaluate centripetal and tangential acceleration in nonuniform circular motion, and find the total acceleration vector.

Uniform circular motion is a specific type of motion in which an object travels in a circle with a constant speed. For example, any point on a propeller spinning at a constant rate is executing uniform circular motion. Other examples are the second, minute, and hour hands of a watch. It is remarkable that points on these rotating objects are actually accelerating, although the rotation rate is a constant. To see this, we must analyze the motion in terms of vectors.

### Centripetal Acceleration

In one-dimensional kinematics, objects with a constant speed have zero acceleration. However, in two- and three-dimensional kinematics, even if the speed is a constant, a particle can have acceleration if it moves along a curved trajectory such as a circle. In this case the velocity vector is changing, or  $d\vec{v}/dt \neq 0$ . This is shown in **Figure 4.18**. As the particle moves counterclockwise in time  $\Delta t$  on the circular path, its position vector moves from  $\vec{r}(t)$  to  $\vec{r}(t + \Delta t)$ . The velocity vector has constant magnitude and is tangent to the path as it changes from  $\vec{v}(t)$  to  $\vec{v}(t + \Delta t)$ , changing its direction only. Since the velocity vector  $\vec{v}(t)$  is perpendicular to the position vector  $\vec{r}(t)$ , the triangles formed by the position vectors and  $\Delta \vec{r}$ , and the velocity vectors and  $\Delta \vec{v}$  are similar. Furthermore, since  $|\vec{r}(t)| = |\vec{r}(t + \Delta t)|$  and  $|\vec{v}(t)| = |\vec{v}(t + \Delta t)|$ , the two triangles are isosceles. From these facts we can make the assertion

$$\frac{\Delta v}{v} = \frac{\Delta r}{r} \text{ or } \Delta v = \frac{v}{r} \Delta r.$$



**Figure 4.18** (a) A particle is moving in a circle at a constant speed, with position and velocity vectors at times  $t$  and  $t + \Delta t$ . (b) Velocity vectors forming a triangle. The two triangles in the figure are similar. The vector  $\Delta \vec{v}$  points toward the center of the circle in the limit  $\Delta t \rightarrow 0$ .

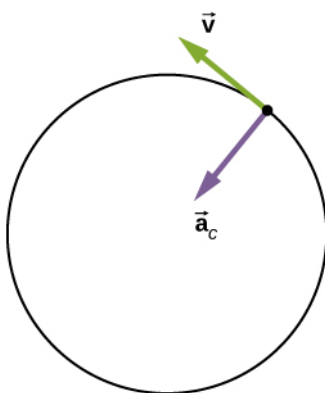
We can find the magnitude of the acceleration from

$$a = \lim_{\Delta t \rightarrow 0} \left( \frac{\Delta v}{\Delta t} \right) = \frac{v}{r} \left( \lim_{\Delta t \rightarrow 0} \frac{\Delta r}{\Delta t} \right) = \frac{v^2}{r}.$$

The direction of the acceleration can also be found by noting that as  $\Delta t$  and therefore  $\Delta \theta$  approach zero, the vector  $\Delta \vec{v}$  approaches a direction perpendicular to  $\vec{v}$ . In the limit  $\Delta t \rightarrow 0$ ,  $\Delta \vec{v}$  is perpendicular to  $\vec{v}$ . Since  $\vec{v}$  is tangent to the circle, the acceleration  $d\vec{v}/dt$  points toward the center of the circle. Summarizing, a particle moving in a circle at a constant speed has an acceleration with magnitude

$$a_C = \frac{v^2}{r}. \quad (4.27)$$

The direction of the acceleration vector is toward the center of the circle (**Figure 4.19**). This is a radial acceleration and is called the **centripetal acceleration**, which is why we give it the subscript  $c$ . The word *centripetal* comes from the Latin words *centrum* (meaning “center”) and *petere* (meaning “to seek”), and thus takes the meaning “center seeking.”



**Figure 4.19** The centripetal acceleration vector points toward the center of the circular path of motion and is an acceleration in the radial direction. The velocity vector is also shown and is tangent to the circle.

Let's investigate some examples that illustrate the relative magnitudes of the velocity, radius, and centripetal acceleration.

### Example 4.10

#### Creating an Acceleration of 1 $g$

A jet is flying at 134.1 m/s along a straight line and makes a turn along a circular path level with the ground. What does the radius of the circle have to be to produce a centripetal acceleration of 1  $g$  on the pilot and jet toward the center of the circular trajectory?

#### Strategy

Given the speed of the jet, we can solve for the radius of the circle in the expression for the centripetal acceleration.

#### Solution

Set the centripetal acceleration equal to the acceleration of gravity:  $9.8 \text{ m/s}^2 = v^2/r$ .

Solving for the radius, we find

$$r = \frac{(134.1 \text{ m/s})^2}{9.8 \text{ m/s}^2} = 1835 \text{ m} = 1.835 \text{ km}.$$

#### Significance

To create a greater acceleration than  $g$  on the pilot, the jet would either have to decrease the radius of its circular trajectory or increase its speed on its existing trajectory or both.



**4.5 Check Your Understanding** A flywheel has a radius of 20.0 cm. What is the speed of a point on the edge of the flywheel if it experiences a centripetal acceleration of  $900.0 \text{ cm/s}^2$ ?

Centripetal acceleration can have a wide range of values, depending on the speed and radius of curvature of the circular path. Typical centripetal accelerations are given in the following table.

Object	Centripetal Acceleration ( $\text{m/s}^2$ or factors of $g$ )
Earth around the Sun	$5.93 \times 10^{-3}$

**Table 4.1 Typical Centripetal Accelerations**

Object	Centripetal Acceleration (m/s <sup>2</sup> or factors of $g$ )
Moon around the Earth	$2.73 \times 10^{-3}$
Satellite in geosynchronous orbit	0.233
Outer edge of a CD when playing	5.78
Jet in a barrel roll	(2–3 $g$ )
Roller coaster	(5 $g$ )
Electron orbiting a proton in a simple Bohr model of the atom	$9.0 \times 10^{22}$

**Table 4.1 Typical Centripetal Accelerations**

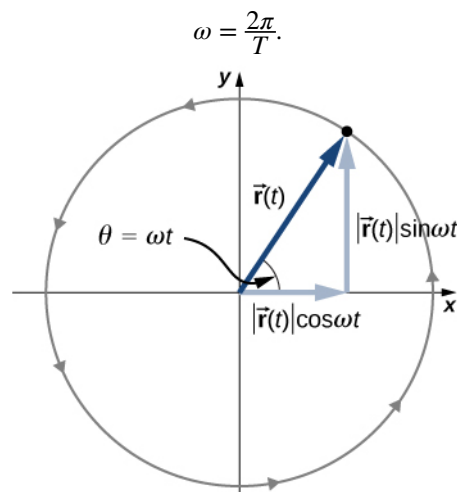
## Equations of Motion for Uniform Circular Motion

A particle executing circular motion can be described by its position vector  $\vec{r}(t)$ . **Figure 4.20** shows a particle executing circular motion in a counterclockwise direction. As the particle moves on the circle, its position vector sweeps out the angle  $\theta$  with the  $x$ -axis. Vector  $\vec{r}(t)$  making an angle  $\theta$  with the  $x$ -axis is shown with its components along the  $x$ - and  $y$ -axes. The magnitude of the position vector is  $A = |\vec{r}(t)|$  and is also the radius of the circle, so that in terms of its components,

$$\vec{r}(t) = A \cos \omega t \hat{i} + A \sin \omega t \hat{j}. \quad (4.28)$$

Here,  $\omega$  is a constant called the **angular frequency** of the particle. The angular frequency has units of radians (rad) per second and is simply the number of radians of angular measure through which the particle passes per second. The angle  $\theta$  that the position vector has at any particular time is  $\omega t$ .

If  $T$  is the period of motion, or the time to complete one revolution ( $2\pi$  rad), then



**Figure 4.20** The position vector for a particle in circular motion with its components along the  $x$ - and  $y$ -axes. The particle moves counterclockwise. Angle  $\theta$  is the angular frequency  $\omega$  in radians per second multiplied by  $t$ .

Velocity and acceleration can be obtained from the position function by differentiation:

$$\vec{v}(t) = \frac{d\vec{r}(t)}{dt} = -A\omega \sin \omega t \hat{i} + A\omega \cos \omega t \hat{j}. \quad (4.29)$$

It can be shown from **Figure 4.20** that the velocity vector is tangential to the circle at the location of the particle, with magnitude  $A\omega$ . Similarly, the acceleration vector is found by differentiating the velocity:

$$\vec{a}(t) = \frac{d\vec{v}(t)}{dt} = -A\omega^2 \cos \omega t \hat{i} - A\omega^2 \sin \omega t \hat{j}. \quad (4.30)$$

From this equation we see that the acceleration vector has magnitude  $A\omega^2$  and is directed opposite the position vector, toward the origin, because  $\vec{a}(t) = -\omega^2 \vec{r}(t)$ .

### Example 4.11

#### Circular Motion of a Proton

A proton has speed  $5 \times 10^6$  m/s and is moving in a circle in the  $xy$  plane of radius  $r = 0.175$  m. What is its position in the  $xy$  plane at time  $t = 2.0 \times 10^{-7}$  s = 200 ns? At  $t = 0$ , the position of the proton is  $0.175 \text{ m } \hat{i}$  and it circles counterclockwise. Sketch the trajectory.

#### Solution

From the given data, the proton has period and angular frequency:

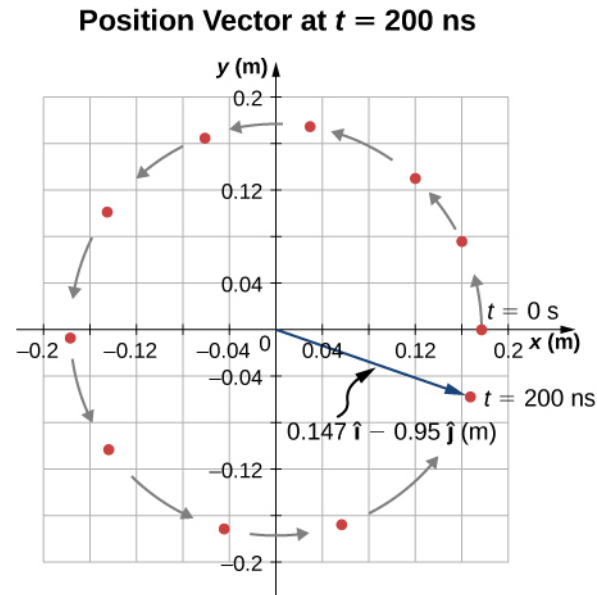
$$T = \frac{2\pi r}{v} = \frac{2\pi(0.175 \text{ m})}{5.0 \times 10^6 \text{ m/s}} = 2.20 \times 10^{-7} \text{ s}$$

$$\omega = \frac{2\pi}{T} = \frac{2\pi}{2.20 \times 10^{-7} \text{ s}} = 2.856 \times 10^7 \text{ rad/s}.$$

The position of the particle at  $t = 2.0 \times 10^{-7}$  s with  $A = 0.175$  m is

$$\begin{aligned} \vec{r}(2.0 \times 10^{-7} \text{ s}) &= A \cos \omega(2.0 \times 10^{-7} \text{ s}) \hat{i} + A \sin \omega(2.0 \times 10^{-7} \text{ s}) \hat{j} \text{ m} \\ &= 0.175 \cos[(2.856 \times 10^7 \text{ rad/s})(2.0 \times 10^{-7} \text{ s})] \hat{i} \\ &\quad + 0.175 \sin[(2.856 \times 10^7 \text{ rad/s})(2.0 \times 10^{-7} \text{ s})] \hat{j} \text{ m} \\ &= 0.175 \cos(5.712 \text{ rad}) \hat{i} + 0.175 \sin(5.712 \text{ rad}) \hat{j} = 0.147 \hat{i} - 0.095 \hat{j} \text{ m}. \end{aligned}$$

From this result we see that the proton is located slightly below the  $x$ -axis. This is shown in **Figure 4.21**.



**Figure 4.21** Position vector of the proton at  $t = 2.0 \times 10^{-7} \text{ s} = 200 \text{ ns}$ . The trajectory of the proton is shown. The angle through which the proton travels along the circle is 5.712 rad, which is a little less than one complete revolution.

### Significance

We picked the initial position of the particle to be on the  $x$ -axis. This was completely arbitrary. If a different starting position were given, we would have a different final position at  $t = 200 \text{ ns}$ .

## Nonuniform Circular Motion

Circular motion does not have to be at a constant speed. A particle can travel in a circle and speed up or slow down, showing an acceleration in the direction of the motion.

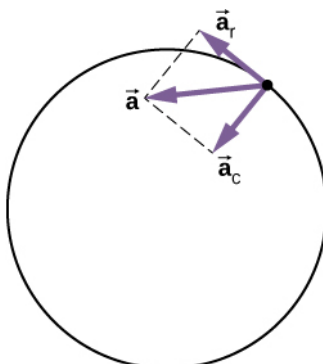
In uniform circular motion, the particle executing circular motion has a constant speed and the circle is at a fixed radius. If the speed of the particle is changing as well, then we introduce an additional acceleration in the direction tangential to the circle. Such accelerations occur at a point on a top that is changing its spin rate, or any accelerating rotor. In **Displacement and Velocity Vectors** we showed that centripetal acceleration is the time rate of change of the direction of the velocity vector. If the speed of the particle is changing, then it has a **tangential acceleration** that is the time rate of change of the magnitude of the velocity:

$$a_T = \frac{d|\vec{v}|}{dt}. \quad (4.31)$$

The direction of tangential acceleration is tangent to the circle whereas the direction of centripetal acceleration is radially inward toward the center of the circle. Thus, a particle in circular motion with a tangential acceleration has a **total acceleration** that is the vector sum of the centripetal and tangential accelerations:

$$\vec{a} = \vec{a}_C + \vec{a}_T. \quad (4.32)$$

The acceleration vectors are shown in **Figure 4.22**. Note that the two acceleration vectors  $\vec{a}_C$  and  $\vec{a}_T$  are perpendicular to each other, with  $\vec{a}_C$  in the radial direction and  $\vec{a}_T$  in the tangential direction. The total acceleration  $\vec{a}$  points at an angle between  $\vec{a}_C$  and  $\vec{a}_T$ .



**Figure 4.22** The centripetal acceleration points toward the center of the circle. The tangential acceleration is tangential to the circle at the particle's position. The total acceleration is the vector sum of the tangential and centripetal accelerations, which are perpendicular.

## Example 4.12

### Total Acceleration during Circular Motion

A particle moves in a circle of radius  $r = 2.0$  m. During the time interval from  $t = 1.5$  s to  $t = 4.0$  s its speed varies with time according to

$$v(t) = c_1 - \frac{c_2}{t^2}, \quad c_1 = 4.0 \text{ m/s}, \quad c_2 = 6.0 \text{ m} \cdot \text{s}.$$

What is the total acceleration of the particle at  $t = 2.0$  s?

### Strategy

We are given the speed of the particle and the radius of the circle, so we can calculate centripetal acceleration easily. The direction of the centripetal acceleration is toward the center of the circle. We find the magnitude of the tangential acceleration by taking the derivative with respect to time of  $|v(t)|$  using **Equation 4.31** and evaluating it at  $t = 2.0$  s. We use this and the magnitude of the centripetal acceleration to find the total acceleration.

### Solution

Centripetal acceleration is

$$v(2.0\text{s}) = \left(4.0 - \frac{6.0}{(2.0)^2}\right) \text{m/s} = 2.5 \text{ m/s}$$

$$a_C = \frac{v^2}{r} = \frac{(2.5 \text{ m/s})^2}{2.0 \text{ m}} = 3.1 \text{ m/s}^2$$

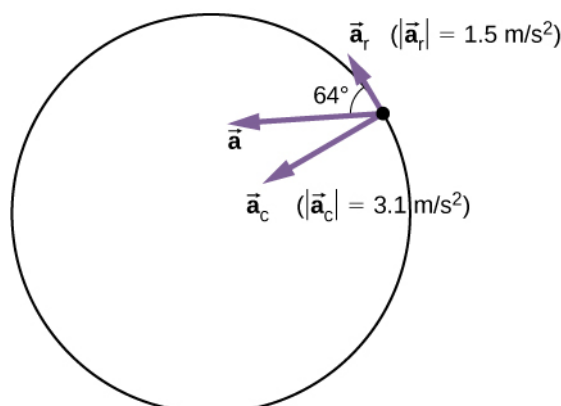
directed toward the center of the circle. Tangential acceleration is

$$a_T = \left| \frac{d\vec{v}}{dt} \right| = \frac{2c_2}{t^3} = \frac{12.0}{(2.0)^3} \text{m/s}^2 = 1.5 \text{ m/s}^2.$$

Total acceleration is

$$|\vec{a}| = \sqrt{3.1^2 + 1.5^2} \text{m/s}^2 = 3.44 \text{ m/s}^2$$

and  $\theta = \tan^{-1} \frac{3.1}{1.5} = 64^\circ$  from the tangent to the circle. See **Figure 4.23**.



**Figure 4.23** The tangential and centripetal acceleration vectors. The net acceleration  $\vec{a}$  is the vector sum of the two accelerations.

### Significance

The directions of centripetal and tangential accelerations can be described more conveniently in terms of a polar coordinate system, with unit vectors in the radial and tangential directions. This coordinate system, which is used for motion along curved paths, is discussed in detail later in the book.

## 4.5 | Relative Motion in One and Two Dimensions

### Learning Objectives

By the end of this section, you will be able to:

- Explain the concept of reference frames.
- Write the position and velocity vector equations for relative motion.
- Draw the position and velocity vectors for relative motion.
- Analyze one-dimensional and two-dimensional relative motion problems using the position and velocity vector equations.

Motion does not happen in isolation. If you're riding in a train moving at 10 m/s east, this velocity is measured relative to the ground on which you're traveling. However, if another train passes you at 15 m/s east, your velocity relative to this other train is different from your velocity relative to the ground. Your velocity relative to the other train is 5 m/s west. To explore this idea further, we first need to establish some terminology.

### Reference Frames

To discuss relative motion in one or more dimensions, we first introduce the concept of **reference frames**. When we say an object has a certain velocity, we must state it has a velocity with respect to a given reference frame. In most examples we have examined so far, this reference frame has been Earth. If you say a person is sitting in a train moving at 10 m/s east, then you imply the person on the train is moving relative to the surface of Earth at this velocity, and Earth is the reference frame. We can expand our view of the motion of the person on the train and say Earth is spinning in its orbit around the Sun, in which case the motion becomes more complicated. In this case, the solar system is the reference frame. In summary, all discussion of relative motion must define the reference frames involved. We now develop a method to refer to reference frames in relative motion.